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Viscous flow near a cusped corner

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The slow motion of fluid exterior to a cylinder lying on a wall is considered for a variety of boundary conditions. In particular, the solution is obtained for the case when the motion far from the cylinder is one of uniform shear. Calculations are made for the force and the moment exerted by the fluid on the cylinder. The asymptotic form of the flow both far from the cylinder and near the cusped corners is presented. The flow sufficiently near a cusp consists of a sequence of eddies of rapidly diminishing strength, and the solution of another boundary-value problem supports the view that the nature of this eddy system is independent of conditions far from the cusp. The nature of inviscid flow with uniform vorticity in cusped corners is also considered.

1. Introduction

The region exterior to a circular cylinder lying on a plane wall, shown in figure 1 (a), is particularly well suited to solving Stokes flow boundary-value problems. The geometry is not only of interest in itself, but has the special feature of cusped corners. Sufficiently near such a cusped corner any plane, incompressible, viscous flow must be slow enough to be governed by the Stokes approximation. Thus the asymptotic forms, near the cusp, of solutions of the Stokes equation in the geometry of figure 1 (a) reveal the nature of the local behaviour of solutions of the Navier–Stokes equations near the corner.

The central boundary-value problem of this paper is the slow motion past a circular cylinder when the flow far from the cylinder is one of uniform shear (figure 2 (a)). With the aid of this solution, the nature of viscous flow in a cusped corner is investigated. The flow consists of a sequence of eddies of rapidly diminishing strength. Moffatt (1964a) considered viscous flow near a sharp corner between two planes on which a variety of boundary conditions were imposed. When both planes were at rest near the corner, and some mechanism far from the corner agitated the fluid, an eddy system was found to exist near the corner. It is reasonable to assume that the nature of viscous flow near a corner is determined only by the geometry and the motion of the local boundaries. The solution of a second boundary-value problem with the geometry of figure 1 (a) will be presented to support this assertion.

That the flow region of figure 1 (a) is suited to solving slow-motion problems derives from the fact that the region can be mapped onto an infinite strip. The transformation

$$w = u + iv = i(x+iy)^{-1}, \tag{1.1}$$

from plane Cartesian co-ordinates (x,y) to co-ordinates (u,v) has the property that, if $\psi(x,y)$ is a solution of the biharmonic equation, $\nabla^4_{x,y}\psi = 0$, then $\Psi(u,v) = (u^2+v^2)\,\psi(x,y)$ is a solution of $\nabla^4_{u,v}\Psi = 0$, where

$$\nabla^4_{x,y} = \frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}, \tag{1.2}$$

$$\nabla^4_{u,v} = \frac{\partial^4}{\partial u^4}+2\frac{\partial^4}{\partial u^2\partial v^2}+\frac{\partial^4}{\partial v^4}. \tag{1.3}$$

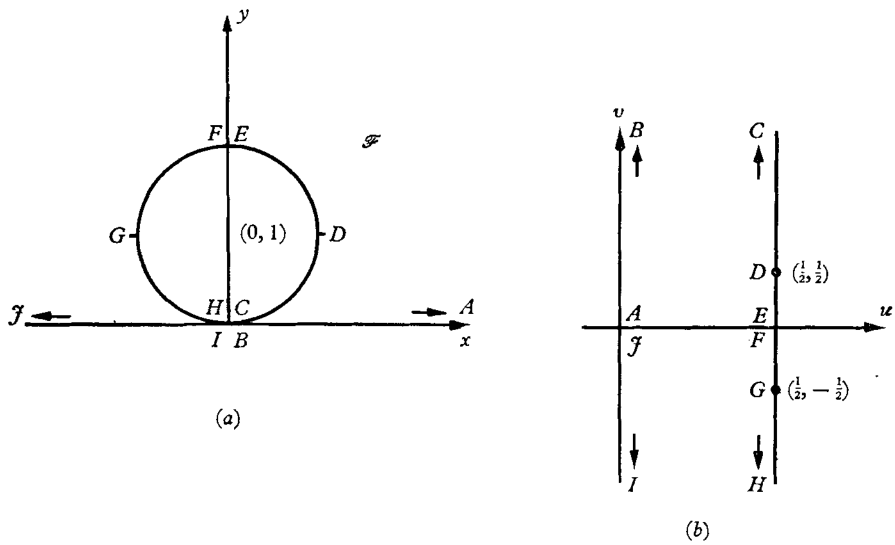

FIGURE 1. (a) The flow region, $\mathscr{F}$, in the (x, y)-plane and (b) its mapping onto the (u, v)-plane.

(1.1) maps the upper half of the (x,y)-plane exterior to the circle

$$y/r^2 = \tfrac{1}{2} \quad (r^2 = x^2+y^2)$$

onto an infinite strip in the (u,v)-plane, lying between $u = 0$ and $u = \frac{1}{2}$ (figure 1).

2. Two simple boundary-value problems

Before considering the problem of shear flow past a circular cylinder, it is worthwhile to look at some simpler problems in which either the cylinder or the wall is in motion near the cusp. These problems provide a means of presenting the mathematical apparatus that will be suitable for solving the more complicated cases considered in this paper. They also provide an interesting contrast to the cases where the boundaries are at rest near the cusp, in that no eddy structure is observed if the boundaries are in motion near the corner. The problems of this section, and others, were studied by Frazer (1926), who used a method different from that employed here.

Sliding wall

Consider first a fixed cylinder and a wall that slides with velocity $h(x)$. It will be noted that unless $h(x) =$ constant the wall stretches as well as slides. We seek a stream function, $\psi(x,y)$, satisfying $\nabla^4_{x,y}\psi = 0$ in the flow region of figure 1(a).

If the x- and y-components of velocity are $-\partial\psi/\partial y$ and $\partial\psi/\partial x$ respectively, then on $y = 0$:

$$\psi = 0, \quad \partial\psi/\partial y = -h(x), \tag{2.1}$$

and on $y/r^2 = \frac{1}{2}$:

$$\psi = \partial\psi/\partial n = 0, \tag{2.2}$$

where $\partial/\partial n$ denotes the normal derivative.

In the (u, v)-plane, with $\Psi = (u^2+v^2)\,\psi$, the problem is $\nabla^4_{u,v}\Psi = 0$ with the boundary conditions that on $u = 0$:

$$\Psi = 0, \quad \partial\Psi/\partial u = -H(v), \tag{2.3}$$

and on $u = \frac{1}{2}$:

$$\Psi = \partial\Psi/\partial u = 0, \tag{2.4}$$

where $H(v) = h(v^{-1})$. The problem will be solved with the aid of a Fourier transform, here defined by

$$\left.\begin{aligned} \hat{\Psi}(u,k) &= \int_{-\infty}^{\infty} e^{-ikv}\Psi(u,v)\,dv, \\ \Psi(u,v) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikv}\hat{\Psi}(u,k)\,dk. \end{aligned}\right\} \tag{2.5}$$

$\hat{\Psi}(u,k)$ is the solution of the equation

$$\frac{\partial^4\hat{\Psi}}{\partial u^4} - 2k^2\frac{\partial^2\hat{\Psi}}{\partial u^2} + k^4\hat{\Psi} = 0, \tag{2.6}$$

with the boundary conditions on $u = 0$:

$$\hat{\Psi} = 0, \quad \partial\hat{\Psi}/\partial u = -\hat{H}(k), \tag{2.7}$$

and on $u = \frac{1}{2}$:

$$\hat{\Psi} = \partial\hat{\Psi}/\partial u = 0. \tag{2.8}$$

The four linearly independent solutions of (2.6) are e^{ku}, e^{-ku}, ue^{ku} and ue^{-ku}. The solution of (2.6) subject to (2.7) and (2.8) is

$$\hat{\Psi} = 2\hat{H}(k)\, G(u,k)/F(k), \tag{2.9}$$

where

$$F(k) = 4\sinh^2\tfrac{1}{2}k - k^2, \tag{2.10}$$

$$G(u,k) = k(\tfrac{1}{2}-u)\sinh ku - 2u\sinh\tfrac{1}{2}k\sinh k(\tfrac{1}{2}-u). \tag{2.11}$$

If the wall slides without stretching, $h(x) = 1$ and, with the aid of Lighthill (1960), $\hat{H}(k) = 2\pi\delta(k)$, where δ denotes the Dirac function. Then $\hat{\Psi}$ vanishes for $k \neq 0$, so that

$$\hat{\Psi} = -8\pi u(u-\tfrac{1}{2})^2\delta(k). \tag{2.12}$$

The stream function is

$$\psi = -4y[(y/r^2)-\tfrac{1}{2}]^2. \tag{2.13}$$

As $r\to\infty$, $\psi\to -y$; i.e. far from the cylinder the motion is that of a uniform stream in the x-direction. As $r\to 0$, $\psi\to 0$, $\partial\psi/\partial x\to 0$ and the value of $\partial\psi/\partial y$ depends on the path on which the origin is approached. The solution (2.9) will also be directly applicable to a problem to be considered in §4.

Rotating cylinder

If the wall is stationary, and the cylinder rotates with constant angular velocity, the motion is determined by solving (2.6) subject to the conditions that on $u = 0$:

$$\hat{\Psi} = \partial\hat{\Psi}/\partial u = 0, \tag{2.14}$$

and on $u = \frac{1}{2}$:

$$\hat{\Psi} = 0, \quad \partial\hat{\Psi}/\partial u = -2\pi\delta(k). \tag{2.15}$$

In the (u, v)-plane, this flow is a simple Couette flow, as is the motion corresponding to (2.13). By a method similar to that above, the solution is found to be

$$\psi = 2y^2r^{-2}(1-2yr^{-2}). \tag{2.16}$$

As $r \to \infty$, $\psi \sim 2y^2/r^2$, so that the velocities tend to zero. As $r \to 0$, ψ and $\partial\psi/\partial x$ tend to zero, and $\partial\psi/\partial y$ approaches a value which is dependent upon the path on which the origin is approached. It should be noted that $y/r^2 \leqslant \frac{1}{2}$ in the flow region.

3. Shear flow past a cylinder

We will investigate the flow of figure 2(*a*). The cylinder and the wall $y = 0$ are at rest, and far from the cylinder the x- and y-components of velocity are y and 0 respectively. χ, the stream function for this motion, is the solution of $\nabla^4_{x,y}\chi = 0$ subject to the boundary conditions that on $y = 0$ and on $y/r^2 = \frac{1}{2}$:

$$\chi = \partial\chi/\partial n = 0, \tag{3.1}$$

and as $r \to \infty$:

$$\chi = -\tfrac{1}{2}y^2 + o(r^2). \tag{3.2}$$

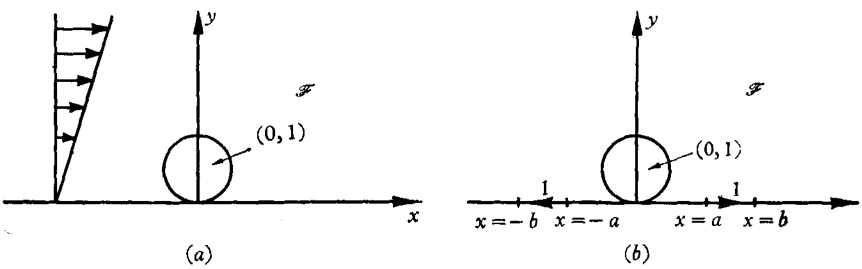

FIGURE 2. Flow about a cylinder tangent to a plane, in which (*a*) far from the cylinder the motion is one of uniform shear and (*b*) the flow is induced by the motion of sleeves in the wall regions $a < x < b$ and $-b < x < -a$.

Alternatively, the problem can be formulated in terms of $\Psi = (\chi/r^2) + \frac{1}{2}(y^2/r^2)$, so that $\hat{\Psi}$ is a solution of (2.6) with the boundary conditions on $u = 0$:

$$\hat{\Psi} = \partial\hat{\Psi}/\partial u = 0, \tag{3.3}$$

and on $u = \frac{1}{2}$:

$$\hat{\Psi} = \tfrac{1}{4}\pi e^{-\frac{1}{2}|k|}, \quad \partial\hat{\Psi}/\partial u = \tfrac{1}{2}\pi(1-\tfrac{1}{2}|k|)e^{-\frac{1}{2}|k|}. \tag{3.4}$$

In terms of Ψ, the boundary-value problem is more straightforward, and leads to the solution

$$\chi = \frac{r^2}{2}\int_{-\infty}^{\infty} \frac{e^{ikv}}{F(k)}G(u,k)\,dk. \tag{3.5}$$

(3.5) also follows directly from (2.9) with

$$h(x) = \tfrac{1}{2}\pi\,\delta(v), \quad \hat{H}(k) = \tfrac{1}{2}\pi, \tag{3.6}$$

which describes the behaviour at $u = 0$, $v = 0$ of the uniform shear whose stream function is $\psi = -\frac{1}{2}y^2$.

The features of (3.5) that are of special interest are the force and moment exerted by the fluid on the cylinder, the nature of the far field ($r \to \infty$) and the motion in the cusped corners ($r \to 0$). The force in the y-direction exerted by the fluid on the cylinder is zero. Since χ is an even function of x, the pressure in the fluid is an odd function of x. Neither the pressure forces nor the viscous forces can exert a net lift on the cylinder.

The force in the x-direction and the moment (about the origin) exerted by the fluid on the cylinder are

$$2\int_0^2 \left\{\frac{y}{x}(y-1)\left(\frac{\partial^3\chi}{\partial y^3}+\frac{\partial^3\chi}{\partial y\,\partial x^2}\right)+y\left(\frac{\partial^3\chi}{\partial x^3}+\frac{\partial^3\chi}{\partial x\,\partial y^2}\right)-2\frac{\partial^2\chi}{\partial x\,\partial y}+\frac{(y-1)}{x}\left(\frac{\partial^2\chi}{\partial x^2}-\frac{\partial^2\chi}{\partial y^2}\right)\right\}dy$$

and

$$2\int_0^2 \left\{\frac{y}{x}(y-1)\left(\frac{\partial^3\chi}{\partial y^3}+\frac{\partial^3\chi}{\partial y\,\partial x^2}\right)+y\left(\frac{\partial^3\chi}{\partial x^3}+\frac{\partial^3\chi}{\partial x\,\partial y^2}\right)+2(1-2y)\frac{\partial^2\chi}{\partial x\,\partial y}+\frac{(2y^2-3y)}{x}\left(\frac{\partial^2\chi}{\partial x^2}-\frac{\partial^2\chi}{\partial y^2}\right)\right\}dy,$$

respectively.

The force and moment are made dimensionless with respect to the dynamic viscosity, the cylinder radius and the mean shear. The integrals are evaluated along the right half of the cylinder. The contribution of the pressure forces to the net force and moment exerted by the fluid on the cylinder are identical. The expressions for the force and moment can be simplified by substituting (3.5) for χ and interchanging the order of integration (this procedure can be rigorously justified). The net force in the x-direction is 4π and the moment is 6π. The force must be applied along the line $y = 1{\cdot}5$ to produce the required moment. The net force and moment exerted by the fluid exterior to the circle $r^2 = 2yc$ $(1 \leqslant c < \infty)$ on the system interior to the circle is independent of c.

If a denotes the cylinder radius, the limit $r/a \to 0$ may be interpreted as one for which the cylinder shrinks to a point singularity, and it is reasonable to assume that the stream function (3.5) has the asymptotic form, as $r \to \infty$,

$$\chi \sim -\tfrac{1}{2}y^2 + rg(\theta) + \sum_{n=0}^{\infty} r^{-n} f_n(\theta), \tag{3.7}$$

where θ is the usual polar co-ordinate defined by $x = r\cos\theta$ and $y = r\sin\theta$. The individual terms of (3.7) must satisfy $\chi = \partial\chi/\partial\theta = 0$ on $\theta = 0$ and on $\theta = \pi$ and be solutions of the biharmonic equation. These considerations lead to the expectation that

$$\chi \sim -\tfrac{1}{2}y^2 + A_0\sin^2\theta + \sum_{n=1}^{\infty} r^{-n}[A_n\sin\theta\sin(n+1)\theta + B_n\{(n+2)\sin n\theta - n\sin(n+2)\theta\}]. \tag{3.8}$$

Starting from (3.5), it is possible to show that the asymptotic form of χ is indeed given by (3.8). Since $G(u,k)/F(k)$ is an even function of k, χ can be determined as an integral over the range 0 to ∞. As $r \to \infty$, $|w| \to 0$ and we seek to expand the integrand of (3.5) in powers of w and $\overline{w}$, where the bar denotes a complex conjugate. Care must be taken in carrying out this expansion so as not to encounter unbounded integrals. The calculation may be organized in the following way. We write

$$G(u,k) = G_1(u,k) + G_2(u,k), \tag{3.9}$$

where

$$G_1(u,k) = \{u(\sinh k - k - 2\sinh^2\tfrac{1}{2}k) + \tfrac{1}{2}k\}\sinh ku - \tfrac{1}{2}uk^2e^{-ku}, \tag{3.10}$$

$$G_2(u,k) = -\tfrac{1}{2}uF(k)e^{-ku}. \tag{3.11}$$

For $k \to 0$, G_1 is analytic and $O(k^4)$ so that G_1/F is analytic.

G_1 is $o(e^k)$ as $k \to \infty$. The contribution of G_2 to the integrand may be integrated explicitly. (3.5) may now be written in the form

$$\chi = -\tfrac{1}{2}y^2 + \mathscr{R}\left\{r^2 \int_0^\infty e^{ikv} \frac{G_1(u,k)}{F(k)}\, dk\right\}. \tag{3.12}$$

As (3.12) clearly shows, the separation of $G(u,k)$ into $G_1(u,k)+G_2(u,k)$ splits the stream function into one for the uniform shear and a stream function for the motion induced by the cylinder. The expansion of (3.12) in powers of w and $\overline{w}$ can now be carried out in a straightforward manner, and, upon returning to the variables (r, θ), we find that

$$\chi = -\tfrac{1}{2}r^2\sin^2\theta + \alpha_1 \sin^2\theta - \frac{\sin^3\theta}{6r}\beta_1 + \sin\theta \sum_{n=2,4,\ldots} r^{-n}\{\sin(n+1)\theta\} \frac{(-1)^{\frac{1}{2}n}}{(n+1)!}\alpha_{n+1}$$
$$+ \sum_{n=3,5,\ldots} r^{-n}\{(n+2)\sin n\theta - n\sin(n+2)\theta\} \frac{(-1)^{\frac{1}{2}(n+1)}\beta_n}{4(n+2)!}, \tag{3.13}$$

where
$$\alpha_n = \int_0^\infty \frac{k^n}{F(k)}(\sinh k - k - 2\sinh^2 \tfrac{1}{2}k + \tfrac{1}{2}k^2)\, dk, \tag{3.14}$$

$$\beta_n = \int_0^\infty \frac{k^{n+3}}{F(k)}\, dk. \tag{3.15}$$

n	$\frac{\alpha_n}{(n+3)!}$	$\frac{\beta_n}{(n+3)!}$
1	0·2418	2·4220
3	0·0782	1·2341
5	0·0522	1·0648
7	0·0415	1·0200
9	0·0352	1·0063

TABLE 1. Coefficients in the far field expansion of the stream function for the shear flow past a cylinder

Bounds on the coefficients α_n and β_n can be determined by observing that, for $k \geqslant 0$, $12(2+k^4) > k^4 e^k/F(k)$ and $e^{-k} \geqslant 1-k+\frac{1}{2}k^2-\frac{1}{6}k^3$. These bounds are $0 < \alpha_n < 7(n+3)!$ and $0 < \beta_n < 13(n+3)!$ for $n = 1, 3, 5, \ldots$, which establishes the convergence of (3.13) for $r > 1$.

The values of $\alpha_n/(n+3)!$ and $\beta_n/(n+3)!$ are given in table 1 for $n = 1, 3, 5, 7$ and 9.

From (3.13) it is clear that the Stokes approximation to the shear flow past a cylinder lying on a wall is not a uniformly valid one. The presence of the wall is sufficient however to determine a unique solution. This is not the case if the cylinder is placed in an unbounded shear flow, as was pointed out by Bretherton (1962).

(3.5) may be evaluated by the theorem of residues to yield a representation of χ which is most convenient for discussing the flow near a cusp. The contour is completed by constructing the remaining three sides of a large square in the half plane $\mathscr{I}(k) \geqslant 0$ if $v \geqslant 0$ or $\mathscr{I}(k) \leqslant 0$ if $v \leqslant 0$. The integrand of (3.5) has simple poles at the zeros of $F(k)$ (although $F(0) = 0$, the point $k = 0$ is not a pole of the integrand). A discussion of the roots of the eigenvalue equation

$$F(k) = 4\sinh^2 \tfrac{1}{2}k - k^2 = 0, \tag{3.16}$$

is given by Buchwald (1964), who provides references to the original papers. If k is a root of (3.16), then so are $-k$, $\bar{k}$ and $-\bar{k}$. Thus locating the roots in the first quadrant suffices to locate them throughout the k-plane.

It is convenient to factor $F(k)$ and consider the equations $\sinh\lambda + \lambda = 0$ and $\sinh\mu - \mu = 0$. λ_n will denote a root of the equation $\sinh\lambda + \lambda = 0$, with positive real and imaginary parts, and μ_n a root of $\sinh\mu - \mu = 0$, also with positive real and imaginary parts. The roots are ordered so that $0 < \mathscr{I}(\lambda_1) < \mathscr{I}(\lambda_2) \ldots$, and $0 < \mathscr{I}(\mu_1) < \mathscr{I}(\mu_2) \ldots$. The roots of (3.16) are $\pm 2\lambda_n$, $\pm 2\bar{\lambda}_n$, $\pm 2\mu_n$, $\pm 2\bar{\mu}_n$. λ_1 to λ_5, and μ_1 to μ_5 inclusive, are listed in table 2. These values were obtained from Buchwald's paper and are presented here for convenience.

n	λ_n	μ_n
1	$2{\cdot}25073 + i4{\cdot}21239$	$2{\cdot}76858 + i7{\cdot}49768$
2	$3{\cdot}10319 + i10{\cdot}7125$	$3{\cdot}35221 + i13{\cdot}9000$
3	$3{\cdot}55108 + i17{\cdot}0734$	$3{\cdot}71677 + i20{\cdot}2385$
4	$3{\cdot}85880 + i23{\cdot}3984$	$3{\cdot}98314 + i26{\cdot}5545$
5	$4{\cdot}09370 + i29{\cdot}7081$	$4{\cdot}19325 + i32{\cdot}8597$

TABLE 2. The first five roots of $\sinh\lambda + \lambda = 0$ and $\sinh\mu - \mu = 0$

The theorem of residues leads to the result that, for $v \geqslant 0$, (3.5) becomes

$$\chi = \mathscr{R}\frac{\pi r^2 i}{4}\sum_{n=1}^{\infty}\left\{\frac{\exp(i2\mu_n v)\,G(u, 2\mu_n)}{\mu_n \sinh^2\frac{1}{2}\mu_n} - \frac{\exp(i2\lambda_n v)\,G(u, 2\lambda_n)}{\lambda_n \cosh^2\frac{1}{2}\lambda_n}\right\}. \tag{3.17}$$

From (3.17), we have, as $v \to \infty$ (we will be interested in the right-hand corner),

$$\chi \sim -\mathscr{R}\frac{\pi r^2 i}{4}\frac{\exp(i2\lambda_1 v)\,G(u, 2\lambda_1)}{\lambda_1\cosh^2\frac{1}{2}\lambda_1}. \tag{3.18}$$

The physical significance of (3.18) will be pointed out in §5.

4. The motion induced by two moving sleeves inserted in the wall

To support the assertion that the character of the flow, sufficiently near the cusps, is independent of the manner in which the fluid is agitated far from the cusps, we investigate the problem depicted in figure 2(*b*). This problem was suggested by the work of Moffatt (1964*b*).

The stream function, ψ, is a solution of $\nabla^4_{x,y}\psi = 0$ and it must satisfy on $y/r^2 = \frac{1}{2}$:

$$\psi = \partial\psi/\partial n = 0, \tag{4.1}$$

and on $y = 0$:

$$\psi = 0, \quad \frac{\partial\psi}{\partial y} = \begin{cases} 0 & (|x| < a, \quad |x| > b, \quad b > a > 0), \\ -1 & (a < x < b), \\ 1 & (-b < x < -a). \end{cases} \tag{4.2}$$

The solution of this problem follows directly from (2.9) by noting that here

$$\hat{H}(k) = \frac{-2}{ik}\left(\cos\frac{k}{a} - \cos\frac{k}{b}\right). \tag{4.3}$$

Then ψ is given by

$$\psi = \frac{-2r^2}{\pi i}\int_{-\infty}^{\infty}\frac{e^{ikv}}{kF(k)}\left(\cos\frac{k}{a} - \cos\frac{k}{b}\right)G(u,k)\,dk. \tag{4.4}$$

This may be evaluated by the theorem of residues. The quantity $(\cos k/a - \cos k/b)$ is written in terms of exponentials, so that the single integral of (4.4) is split into the sum of four integrals, the integrands of which are

$$\exp\{ik[v+(1/a)]\}, \quad \exp\{ik[v-(1/a)]\}, \quad -\exp\{ik[v+(1/b)]\},$$

and

$$-\exp\{ik[v-(1/b)]\}$$

each multiplied by the same function of u and k. The contours for these four integrals are each closed by a large square in that half of the k-plane that prevents the exponential factor from approaching infinity as the size of the square becomes indefinitely large. The poles of the integrand of (4.4) are identical to those of the integrand of (3.5). Application of the residue theorem to (4.4) leads to the result that for $v > 1/a$

$$\psi = \mathscr{R}\frac{r^2}{2}\sum_{n=1}^{\infty}\frac{\exp(i2\lambda_n v)\,G(u,2\lambda_n)}{\lambda_n^2\cosh^2\frac{1}{2}\lambda_n}\left(\cos\frac{2\lambda_n}{a} - \cos\frac{2\lambda_n}{b}\right) + \frac{\exp(i2\mu_n v)\,G(u,2\mu_n)}{\mu_n^2\sinh^2\frac{1}{2}\mu_n}\left(\cos\frac{2\mu_n}{b} - \cos\frac{2\mu_n}{a}\right). \tag{4.5}$$

The flow near the right-hand corner is given by the asymptotic form of (4.5) as $v \to \infty$, which is

$$\psi \sim \mathscr{R}\frac{r^2}{2}\frac{\exp(i2\lambda_1 v)\,G(u,2\lambda_1)}{\lambda_1^2\cosh^2\frac{1}{2}\lambda_1}\left(\cos\frac{2\lambda_1}{a} - \cos\frac{2\lambda_1}{b}\right). \tag{4.6}$$

5. Flow near a cusped corner

As is indicated by (3.18) and (4·6), sufficiently near the cusped corner, the stream function, ψ, of any viscous incompressible flow must be given by

$$\psi \sim \mathscr{R}Kr^2\exp(i2\lambda_1 v)\,G(u,2\lambda_1), \tag{5.1}$$

where K is a complex constant depending on λ_1 and conditions far from the corner. If we approach the origin of the (x,y)-plane by moving along the circle

$$y/r^2 = 1/2c = u \quad (c > 1 \quad \text{and} \quad x \geqslant 0),$$

then from (5.1) with $\lambda_1 = \xi_1 + i\eta_1$, ψ behaves like $r^2\exp(-2\eta_1 x/r^2)$ (except for multiplicative factors of $O(1)$). The ratio of the neglected inertia terms to the viscous terms retained by the Stokes approximation provides the criterion for determining the region of validity of the approximation. This criterion is

$$R = r^2\exp[-2\eta_1 x/r^2] \ll 1,$$

where R is the Reynolds number based on the radius of the circular cylinder and the characteristic velocity in the problem (the uniform shear times the cylinder radius).

(5.1) describes the flow near the corner as a sequence of eddies of diminishing strength. If $v = V(u)$ is the equation of a curve on which $\psi = 0$, then ψ is also zero on the curve

$$v = V(u) + \frac{n\pi}{2\xi_1} \quad (n = 1, 2, 3, \ldots).$$

The cusp region, $v \to \infty$, is divided up into a repetitive cell pattern, the boundaries of each cell being curves on which $\psi = 0$. Enclosed within each cell is an eddy. The situation is depicted in figure 3. Let Ψ_n denote the value of the transformed stream function, Ψ, at the centre of the nth cell (under some arbitrary system of labelling in which n increases with v). Then $r^2\Psi_n$ represents the flux between the

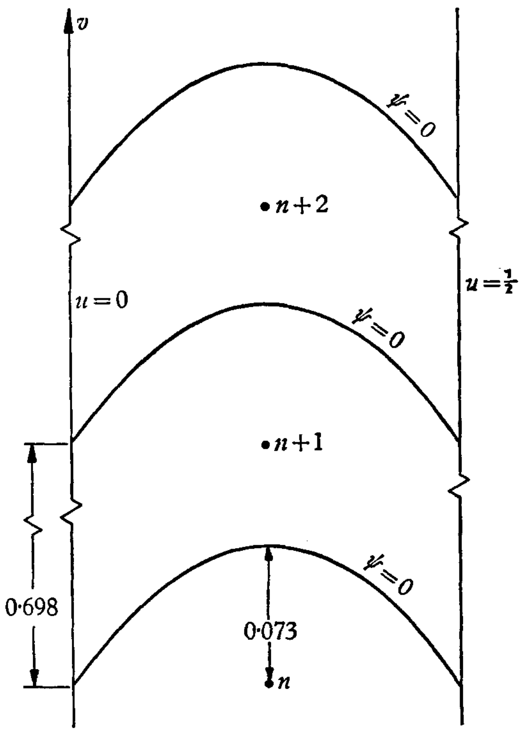

FIGURE 3. The sequence of eddy-containing cells in the region $v \to \infty$.

boundary and the centre of the nth cell, and the ratio $|\Psi_{n+1}/\Psi_n|$ is a convenient measure of the intensity of consecutive eddies. From (5.1), this ratio is

$$\exp(-\pi\eta_1/\xi_1).$$

The intensities of consecutive eddies form a geometric progression with the ratio $e^{-5.87969}$.

6. Flow with uniform vorticity in a cusped corner

The nature of inviscid, incompressible flow with uniform vorticity near a sharp corner was investigated by Fraenkel (1961). In this section we extend that work to include the flow near a cusped corner. The solutions to be presented here contrast interestingly with those of the previous sections, in that no sequence of corner eddies is found for flows with uniform vorticity.

The region of interest is $r \to 0$, $0 \leqslant y \leqslant Ax^\nu$, $x \geqslant 0$, $\nu > 1$; the stream function satisfies $\nabla^2_{x,y}\psi = -1$ within this region, and is zero on the boundaries. The function

$$\psi = -\frac{1}{2}\left\{y^2 - \frac{A}{\nu+1} r^{\nu+1} \sin(\nu+1)\theta\right\} \tag{6.1}$$

satisfies the differential equation and is zero on $\theta = 0$. It is also zero on the curve whose equation is

$$r^{\nu-1} = \frac{(\nu+1)}{A} \frac{\sin^2\theta}{\sin(\nu+1)\theta}. \tag{6.2}$$

(6.2) may be rewritten in the form

$$y = Ax^\nu \frac{\sin(\nu+1)\theta}{(\nu+1)(\sin\theta)(\cos\theta)^\nu},$$

from which it is clear that, as $\theta \to 0$, (6.2) becomes $y = Ax^\nu$.

The solution (6.1) is not unique. To it may be added an arbitrary multiple of

$$\psi = \exp\left\{\frac{-\pi\cos(\nu-1)\theta}{A(\nu-1)r^{\nu-1}}\right\}\sin\left\{\frac{\pi\sin(\nu-1)\theta}{A(\nu-1)r^{\nu-1}}\right\}. \tag{6.3}$$

(6.3) is a solution of Laplace's equation and is zero on $\theta = 0$, and on a curve that approaches $y = Ax^\nu$ as $x \to 0$. The local behaviour of the potential flow, (6.3), is seen to be transcendentally weak compared to the rotational solution, (6.1).

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